

# ON THE GENESIS OF A DETERMINANTAL IDENTITY

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*This paper is dedicated to V. S. Sunder on his 60th birthday.*

ABSTRACT. We show how to derive a  $3 \times 3$  determinantal identity in 12 indeterminates that gives an explicit version of a result of Mohan Kumar and Pavaman Murthy on completing unimodular rows.

## 1. INTRODUCTION

Let  $A$  be a commutative ring with identity. A sequence  $a_1, a_2, \dots, a_n$  of elements of  $A$  is said to be a unimodular sequence if the ideal  $(a_1, \dots, a_n)A = A$ . Given such a unimodular sequence, there is a projective (in fact, stably free)  $A$ -module, say  $P$ , associated with it and defined by the split short exact sequence:

$$0 \longrightarrow A \xrightarrow{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}} A^n \longrightarrow P \longrightarrow 0.$$

It is well known and easy to see that  $P$  is free exactly when the column matrix above can be completed to an invertible matrix over  $A$ , in which case the unimodular sequence is said to be completable.

It is an interesting problem to find conditions under which a unimodular sequence is completable. The first such non-trivial result was by Swan and Towber in [SwnTwb1975] who showed that if  $a, b, c$  is unimodular then  $a^2, b, c$  is completable. In particular, they showed that this follows (easily) from the determinantal identity

$$\det \begin{bmatrix} a^2 & b + ar & c - aq \\ b & -r^2 + bpr & p + qr + cpr \\ c & -p + qr - bpq & -q^2 - cpq \end{bmatrix} = (pa + qb + rc)^2,$$

that holds for commuting indeterminates  $a, b, c, p, q, r$ .

This result of Swan and Towber had a beautiful generalisation due to Suslin in [Ssl1977] who showed that if  $a_0, a_1, \dots, a_n$  is unimodular and  $r_0, r_1, \dots, r_n \in \mathbb{N}$ , then  $a_0^{r_0}, a_1^{r_1}, a_2^{r_2}, \dots, a_n^{r_n}$  is completable provided  $n!$  divides  $r_0 r_1 \dots r_n$ . While Suslin's proof is, in principle, constructive and yields a completion of the unimodular row, I am not aware of a determinantal identity that implies it. Suslin's result was generalised conjecturally by Nori and a special case of this conjecture was proved by Mohan Kumar in [Kmr1997].

The main result of [KmrMrt2010] is another such condition that applies to a length 3 unimodular sequence  $a, b, c$  and states that it is completable if  $z^2 + bz + ac = 0$  has its roots in  $A$ . It is natural to ask whether this result is an easy consequence of some determinantal identity and the main observation of this paper is an affirmative answer to this question.

## 2. THE DETERMINANTAL IDENTITY

The following determinantal identity can be checked directly by a computation or more easily using a computer algebra system such as *Macaulay 2*.

**Proposition 1.** *For indeterminates  $g, h, j, k, s, t, u, v, w, x, y, z$ , the following determinantal identity holds.*

$$\det \begin{bmatrix} gj & g^2j^2x + h^2j^2z - h j v & g^2k^2x + h^2k^2z + h j u \\ gk + h j & g j t - h k v & h k u - g j s \\ h k & g^2j^2w + h^2j^2y + g k t & g^2k^2w + h^2k^2y - g k s \end{bmatrix} = \\ (g^2j^2s + g^2k^2t + h^2j^2u + h^2k^2v)(g^2j^2w + g^2k^2x + h^2j^2y + h^2k^2z).$$

An immediate corollary of Proposition 1 is the following theorem.

**Theorem 2** (Theorem 3 of [KmrMrt2010]). *Let  $A$  be a ring and  $a, b, c \in A$  be a unimodular sequence such that  $z^2 + bz + ac = 0$  has its roots in  $A$ . Then,  $a, b, c$  is completable.*

*Proof.* Let  $\alpha, \beta \in A$  be the roots of  $z^2 + bz + ac = 0$ , so that  $\alpha + \beta = -b$  and  $\alpha\beta = ac$ . Consider now the polynomial ring  $\mathbb{Z}[g, h, j, k]$  and its subring  $\mathbb{Z}[gj, gk, hj, hk]$ , which is easily seen to be isomorphic to  $\mathbb{Z}[W, X, Y, Z]/(WZ - XY)$ . There is thus a ring homomorphism from  $\mathbb{Z}[gj, gk, hj, hk]$  to  $A$  taking  $gj$  to  $a$ ,  $gk$  to  $-\alpha$ ,  $hj$  to  $-\beta$  and  $hk$  to  $c$ .

Now since  $a, b, c$  is unimodular, so is  $a, \alpha, \beta, c$  and so also is  $a^2, \alpha^2, \beta^2, c^2$ . Thus choose  $s, t, u, v, w, x, y, z \in A$  such that  $a^2s + \alpha^2t + \beta^2u + c^2v = 1 = a^2w + \alpha^2x + \beta^2y + c^2z$ . Finally, the image of the  $3 \times 3$  matrix of Proposition 1 under the natural homomorphism to  $A$  is seen to have determinant 1 and first column  $a, b, c$ , thus yielding an explicit completion of the unimodular sequence.  $\square$

## 3. DERIVATION OF THE IDENTITY

While the proof of Theorem 2 is mathematically complete and gives a third proof of the main result of [KmrMrt2010] (their paper has two proofs), it is unsatisfying without an explanation of the genesis of the determinantal identity used. In this section, we remedy this defect by explaining how such an identity was arrived at using the computer algebra system *Macaulay 2*. Thus some arguments in this section depend on the output of calculations of *Macaulay 2* and cannot be regarded as proofs in the mathematical sense.

We begin by recalling the second proof of Theorem 2 from [KmrMrt2010]. Let  $A$  be the ring  $\mathbb{Z}[a, b, c, d, p, q, r]/(ad - bc, pa + q(b + c) + rd - 1)$  and let  $P$  be the stably-free  $A$  module associated to the unimodular sequence  $a, b + c, d$ . It suffices to see that  $P$  is free. Since  $P$  is of rank 2, to show that  $P$  is free, it suffices to map it onto a 2-generated invertible ideal, say  $I$ , of  $A$ . For then, the kernel of this map is another invertible ideal, say  $J$ , of  $A$ . Since  $\det(P) \cong A$  (by stable freeness), it follows that  $IJ$  is principal and so  $J \cong I^{-1}$ . Thus  $P \cong I \oplus I^{-1}$  which is free by a lemma of Swan. They then show that  $P$  maps onto the invertible ideal  $I = (a, b)^2A$ . What we will do is to make each step and map in their proof as explicit as possible. We make our first appeal to *Macaulay 2* to check that  $A$  is an integral domain.

They first observe that the ideal  $(a, b)A$  is an invertible (equivalently projective) ideal, for, localised at a maximal ideal not containing either  $a$  or  $b$ , it becomes the unit ideal, while localising at a maximal ideal not containing either  $c$  or  $d$  makes it principal and generated by  $b$  or  $a$  respectively. However by unimodularity, no maximal ideal contains

all of  $a, b, c, d$ . Next they note that  $(a, b)^2 A = (a^2, b^2) A$  since this is true locally. Alternatively, we may note that multiplying the relation  $pa + q(b + c) + rd = 1$  by  $ab$  and using that  $ad = bc$  gives  $ab = a^2(pb + qd) + b^2(qa + rc)$ .

Next, we see that the inverse of the invertible ideal  $(a, b)A$  is  $(a, c)A$ . For, their product is  $(a^2, ab, ac, bc)A = (a^2, ab, ac, ad)A = a(a, b, c, d)A = aA \cong A$ . Thus the inverse of  $I = (a^2, b^2)A$  is  $J = (a^2, c^2)A$  and their product is  $a^2 A$ .

We know now, in principle, that  $A^2 \cong I \oplus J$ . What we seek is an explicit exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A^2 \xrightarrow{\pi} I \longrightarrow 0,$$

together with a retract, say  $\rho$ , of  $\iota$ . The surjection  $\pi : A^2 \rightarrow I$  is easy to describe. With  $e_1, e_2$  denoting the standard basis vectors in  $A^2$ , set  $\pi(e_1) = a^2$  and  $\pi(e_2) = b^2$ . Let  $K = \ker(\pi)$ .

Now invoke *Macaulay 2* to see that  $K$  is the submodule of  $A^2$  generated by the columns of the matrix

$$\begin{bmatrix} -b^2 & -d^2 \\ a^2 & c^2 \end{bmatrix}.$$

There is a well-defined (and clearly injective) map  $J \rightarrow K$  defined by  $x \mapsto ye_1 + xe_2$  where  $d^2x + c^2y = 0$ . It is easily checked that this map is an isomorphism. Thus the map  $\iota : J \rightarrow A^2$  is given by  $\iota(x) = ye_1 + xe_2$  with  $y$  as above. Suppose that  $\rho : A^2 \rightarrow J$  is a splitting of this map. Let  $\rho(e_1) = -(ta^2 + vc^2)$  and  $\rho(e_2) = sa^2 + uc^2$ . A simple calculation now shows that a necessary and sufficient condition for  $\rho \circ \iota$  to be  $id_J$ , is that  $sa^2 + tb^2 + uc^2 + vd^2 = 1$ . We know that such  $s, t, u, v$  certainly exist in  $A$  so choose them. The map  $\rho \oplus \pi : A^2 \rightarrow J \oplus I$  is then an isomorphism.

The next step in the analysis is to construct an explicit isomorphism from  $J \oplus I$  to  $P$ . Since  $P$  is, by definition, the quotient of  $A^3$  by the cyclic submodule generated by  $ae_1 + (b+c)e_2 + de_3$  (where  $e_1, e_2, e_3$  are the standard basis vectors of  $A^3$ ), the surjection  $A^3 \rightarrow I$  defined by  $e_1 \mapsto b^2$ ,  $e_2 \mapsto -ab$  and  $e_3 \mapsto a^2$  factors through  $P$ , say as  $\tilde{\pi} : P \rightarrow I$ . *Macaulay 2* now shows that the kernel, say  $L$ , of  $A^3 \rightarrow I$  is generated by the columns of the matrix:

$$\begin{bmatrix} c & -a & 0 & 0 \\ d & -b & -c & a \\ 0 & 0 & -d & b \end{bmatrix}.$$

(Actually, *Macaulay 2* gives 14 generators but inspection shows that all are contained in the submodule generated by these 4). Naturally  $L$  contains the vector  $ae_1 + (b+c)e_2 + de_3$  and the quotient of  $L$  by the cyclic submodule generated by this vector (which is a direct summand of  $A^3$  and hence also of  $L$ ) is the kernel of  $\tilde{\pi}$ . In other words,  $\ker(\tilde{\pi})$  is generated by  $c\bar{e}_1 + d\bar{e}_2$ ,  $a\bar{e}_1 + b\bar{e}_2 = -c\bar{e}_2 - d\bar{e}_3$  and  $a\bar{e}_2 + b\bar{e}_3$ . We need to explicitly identify this submodule of  $P$  with  $J$ .

Now  $J$  is generated by  $c^2, a^2$  subject only to the relations given by the columns of the matrix

$$\begin{bmatrix} a^2 & b^2 \\ -c^2 & -d^2 \end{bmatrix}.$$

Thus there is a well-defined map  $\tilde{\iota} : J \rightarrow P$  given by  $\tilde{\iota}(c^2) = c\bar{e}_1 + d\bar{e}_2$ ,  $\tilde{\iota}(a^2) = -a\bar{e}_2 - b\bar{e}_3$ . Using that  $ac = a^2(pc + qd) + c^2(qa + rb)$ , we see that  $\tilde{\iota}(ac) = a\bar{e}_1 + b\bar{e}_2$ , and so  $\tilde{\iota}(J) = \ker(\tilde{\pi})$ . Since we know that  $\ker(\tilde{\pi}) \cong J$  and a surjective endomorphism of a finitely generated module (over any commutative ring) is an isomorphism,  $\tilde{\iota}$  is an isomorphism of  $J$  onto  $\ker(\tilde{\pi})$ .

Thus we now have an exact sequence

$$0 \longrightarrow J \xrightarrow{\tilde{\iota}} P \xrightarrow{\tilde{\pi}} I \longrightarrow 0,$$

and we seek a section, say  $\tilde{\sigma} : I \rightarrow P$ , of  $\tilde{\pi}$ . Since this seems difficult to split directly, we consider the above exact sequence tensored with  $J$  which gives

$$0 \longrightarrow J^2 \xrightarrow{\hat{\iota}} JP \xrightarrow{\hat{\pi}} A \longrightarrow 0,$$

which certainly is easier to split. Here  $\hat{\iota} = \tilde{\iota}|_{J^2}$  while  $\hat{\pi} = a^{-2}\tilde{\pi}|_{JP}$ . Note that under  $\hat{\pi}$ ,  $a^2\bar{e}_1 \mapsto b^2$ ,  $a^2\bar{e}_3 \mapsto a^2$ ,  $c^2\bar{e}_1 \mapsto d^2$  and  $c^2\bar{e}_3 \mapsto c^2$ . Thus a splitting of  $\hat{\pi}$  is given by the map  $\hat{\sigma} : A \rightarrow JP$  given by  $\hat{\sigma}(1) = xa^2\bar{e}_1 + wa^2\bar{e}_3 + zc^2\bar{e}_1 + yc^2\bar{e}_3$ , where  $w, x, y, z$  satisfy  $wa^2 + xb^2 + yc^2 + zd^2 = 1$ . Now tensor with  $I$  to get a splitting of our original sequence. Thus,  $\tilde{\sigma}$  is specified by

$$\begin{aligned} \tilde{\sigma}(a^2) &= a^2(x\bar{e}_1 + w\bar{e}_3) + c^2(z\bar{e}_1 + y\bar{e}_3), \quad \text{and} \\ \tilde{\sigma}(b^2) &= b^2(x\bar{e}_1 + w\bar{e}_3) + d^2(z\bar{e}_1 + y\bar{e}_3). \end{aligned}$$

Finally conclude that the map  $\tilde{\iota} + \tilde{\sigma} : J \oplus I \rightarrow P$  is an isomorphism.

Composing the isomorphisms of  $A^2$  to  $J \oplus I$  and  $J \oplus I$  to  $P$  we get an explicit isomorphism  $A^2 \rightarrow P$  given by:

$$\begin{aligned} e_1 &\mapsto (a^2x + c^2z - cv)\bar{e}_1 + (at - dv)\bar{e}_2 + (a^2w + c^2y + bt)\bar{e}_3 \\ e_2 &\mapsto (b^2x + d^2z + cu)\bar{e}_1 + (du - as)\bar{e}_2 + (b^2w + d^2y - bs)\bar{e}_3 \end{aligned}$$

It follows that an explicit completion of the unimodular sequence  $a, b + c, d$  is given by the matrix:

$$\begin{bmatrix} a & a^2x + c^2z - cv & b^2x + d^2z + cu \\ b + c & at - dv & du - as \\ d & a^2w + c^2y + bt & b^2w + d^2y - bs \end{bmatrix}.$$

And indeed, when we calculate the determinant of the above matrix in the ring  $A$ , it is 1.

Noticing that the elements  $p, q, r$  of  $A$  do not occur in the above matrix we hence try to compute the determinant in the ring  $\mathbb{Z}[a, b, c, d, s, t, u, v, w, x, y, z]/(ad - bc)$ . The answer turns out to be  $(a^2s + b^2t + c^2u + d^2v)(a^2w + b^2x + c^2y + d^2z)$ . Since the subring  $\mathbb{Z}[a, b, c, d]$  of this ring is isomorphic to  $\mathbb{Z}[gj, gk, hj, hk]$ , it follows that the determinantal identity of Proposition 1 holds.

#### ACKNOWLEDGMENTS

I am extremely grateful to Prof. T. Y. Lam for sparking my interest in problems of this kind and to Prof. Pavaman Murthy for encouraging me to write this up.

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